Numerieke wiskunde 2, WINM2-08 2011/12 semester II a

Examination, April 2nd, 2012.

Name	**Student number**

Notes:

- You may use one sheet (single side written) with notes from the lectures.
- During the exam it is NOT permitted to consult books, handouts, other notes.
- Numerical/graphic calculators are permitted, programmable calculators are NOT permited.
- Devices with wireless internet connection and/or document readers are NOT permitted.
- Hint: please describe the solution procedures in full details, not only the results.
- **Normering:** to pass the exam, You need to gather at least half of the total points at the final exam. The final grade for the course is computed by averaging the grade for the practicals (40%) and the grade for the exam (60%).

TEST (to be returned by 17:00)

1. *Complexity of the Gaussian elimination algorithm.*

   Consider a system of linear equations $Ax = b$, where A is a complex matrix of dimension $n \times n$ and b is a complex vectors of size n.

   (a) **[pts 4]** Convert this problem into the equivalent problem of solving a real square linear system of dimension $2n$. Write $A = A_1 + iA_2$, $b = b_1 + ib_2$, $x = x_1 + ix_2$, with A_1, A_2, b_1, b_2, x_1, x_2 all real quantities. Determine the equations that are satisfied by x_1 and x_2.

   (b) **[pts 6]** Determine the number of operations and the computer memory needed by the Gaussian elimination algorithm to solve the complex linear system $Ax = b$ using the method described in (a). Compare these results with the number of operations and the memory needed to solve $Ax = b$ using Gaussian elimination and complex arithmetic.

   Solution.

   (a) In partitioned form,

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(b) Let system 1 denote the real system of part (a), and let system 2 denote the original complex system $Ax = b$. For the matrix storage requirements, system 1 requires $4n^2$ locations, and system 2 requires $2n^2$ locations (each complex number requires two storage locations). To solve system 1 requires about $\frac{1}{3}(2n)^3 = \frac{8}{3}n^3$ multiplications and divisions. System 2 requires $\frac{1}{3}n^3$ complex multiplications and divisions. Since each complex multiplication requires four real multiplications, the actual operation count is $\frac{4}{3}n^3$. Thus system 2 requires half the storage requirements and about half the operation time of system 1.

2. *Choleski decomposition.*

In the course, we have studied that a symmetric positive definite matrix always admits a Choleski factorization of the form $A = LL^T$, where L is a lower triangular matrix with positive diagonal elements.

(a) **[pts 4]** Prove that the viceversa of the statement is also true. More precisely, prove that if $A = LL^T$ with L real and nonsingular, then A is symmetric and positive definite.

(b) **[pts 5]** Using the Choleski method, calculate the decomposition $A = LL^T$ for the matrix

$$\begin{bmatrix} 2.25 & -3.0 & 4.5 \\ -3.0 & 5.0 & -10.0 \\ 4.5 & -10.0 & 34.0 \end{bmatrix}.$$

Solutions. $(Ax, x) = (LL^T x, x) = (L^T x, L^T x) = \|L^T x\|_2^2 > 0$ for all $x \neq 0$, since L^T is nonsingular. Also $det(A) = det(L)^2$.

3. *Orthogonal polynomials*

Let the polynomials $\varphi_j, j = 0, 1, \ldots,$ form an orthogonal system on the interval $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$.

(a) **[pts 4]** Show that the polynomials $\varphi_j((2x - a - b)/(b - a))$, $j = 0, 1, \ldots,$ represent an orthogonal system in the interval $[a, b]$ with respect to the same weight function.

(b) **[pts 5]** From the Legendre polynomials

$$\varphi_0(x) \equiv 1,$$
$$\varphi_1(x) \equiv x,$$
$$\varphi_2(x) \equiv x^2 - \tfrac{1}{3},$$

defined in the interval $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$, obtain the orthogonal polynomials $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$ defined in the interval $[0, 1]$ with respect to the same weight function.

Solutions.

(a) Since the polynomials $\varphi_j(x)$ are orthogonal on the interval $[-1, 1]$, we know that

$$\int_{-1}^{1} \varphi_i(t)\varphi_j(t)dt = 0, i \neq j.$$

In this integral, we make the change of variable

$$t = (2x - a - b)/(b - a), \quad x = \frac{1}{2}[(b - a)t + a + b],$$

and it becomes

$$\int_{a}^{b} \varphi_i((2x - a - b)/(b - a))\varphi_j((2x - a - b)/(b - a))\frac{2}{b - a}dx.$$

This shows that the new polynomials form an orthogonal system on the interval $[a, b]$.

(b) From the Legendre polynomials,

$$\begin{aligned}
\phi_0(t) &= 1 \\
\phi_1(t) &= t \\
\phi_2(t) &= t^2 - 1/3
\end{aligned}$$

we write $t = 2x - 1$ and get the orthogonal polynomials on $[0, 1]$ in the form

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= 2x - 1 \\
\phi_2(x) &= (2x - 1)^2 - 1/3 = 4x^2 - 4x + 2/3
\end{aligned}$$

Normalising each of the Legendre polynomials so that its value at $x = 1$ is equal to 1, the polynomials $\phi_0, \phi_1, \phi_2, \phi_3$ are obtained

$$\begin{aligned}
\phi_0(x) &= 1, \\
\phi_1(x) &= x, \\
\phi_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2},
\end{aligned}$$

These are the first four elements of the system of Legendre polynomials, orthogonal on the interval $(-1, 1)$ with respect to the weight function $w(x) \equiv 1$.

4. *Householder and Givens transformations.*

(a) **[pts 5]** Using Householder reductions, compute the QR factors of the matrix

$$A = \begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix}.$$

(b) **[pts 5]** Compute the QR factors of the same matrix of part (a) using Givens reductions.

Solutions.

(a) Householder reduction produces

$$R_2R_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & 4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 19 & -34 \\ -2 & -5 & 2- \\ 2 & 8 & 37 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix} = R$$

so

$$Q = (R_2R_1)^T = \begin{pmatrix} 1/3 & 14/15 & -2/15 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & -2/15 & 11/15 \end{pmatrix}.$$

(b) Givens reduction produces $P_{23}P_{13}P_{12}A = R$, where

$$P_{12} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_{13} = \begin{pmatrix} \sqrt{5}/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{pmatrix}$$
$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 11/5\sqrt{5} & -2/5\sqrt{5} \\ 0 & 2/5\sqrt{5} & 11/5\sqrt{5} \end{pmatrix}$$

5. *Eigenvalues.*

[pts 5] Explain why the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 12 & 0 & -4 \\ 1 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix}$$

must have at least two real eigenvalues.

Solutions. Since one Geschgorin circle (derived from row sums and shown below) is isolated from the union of the other three circles, Gerschgorin theorems insure that there is one eigenvalue in the isolated circle and three eigenvalues in the union of the other three. But the eigenvalues of real matrices occur in conjugate pairs. So, the root in the isolated circle must be real and there must be at least one real root in the union of the other three circles. Computation reveals that $\sigma(A) = \{\pm i, 2, 10\}$.

6. *Polynomial of best approximation to functions.*

[pts 7] Construct the polynomial of best approximation p_1 of degree one, defined on the interval $[-2, 1]$, for the function f defined by $f(x) = |x|$.

Solution. The minimax polynomial must be such that $f(x) - p_1(x)$ has three alternating extrema in $[-2, 1]$ due to the Oscillating Theorem. Since f is convex, two of these extrema are at the ends -2 and 1, and the other must clearly be at 0. Graphically, the line p_1 must be parallel to the chord joining $(-2, f(-2))$ and $(1, f(1))$. Thus

$$p_1(x) = c_0 - \frac{1}{3}x.$$

The alternating extrema are then

$$\begin{aligned}
&f(-2) - p_1(-2) = 2 - \left(c_0 + \tfrac{2}{3}\right) = \tfrac{4}{3} - c_0 \\
&f(0) - p_1(0) = 0 - (c_0) = -c_0 \\
&f(1) - p_1(1) = 1 - \left(c_0 - \tfrac{1}{3}\right) = \tfrac{4}{3} - c_0.
\end{aligned}$$

These have the same magnitude if

$$\frac{4}{3} - c_0 = -(-c_0),$$

so that the minimax polynomial is

$$p_1(x) = \frac{2}{3} - \frac{1}{3}x,$$

and

$$\|f - p_1\|_\infty = \frac{2}{3}.$$